# The Cauchy problem for discrete time fractional evolution equations*

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**Abstract**

This paper is devoted to the study of discrete-time fractional evolution equations involving the Riemann-Liouville-like difference operator. Based on the relationship between $C_0$-semigroups and a distinguished class of sequences of operators, we show the structure of the solutions for the inhomogenous Cauchy problem of abstract fractional difference equations. Further, we establish two criteria for the existence and uniqueness of solutions for the semilinear Cauchy problem. Some examples are also provided to illustrate our main results.

**Keywords:** Fractional difference equations; $C_0$-semigroups; $\alpha$-resolvent sequences; Inhomogeneous discrete-time fractional Cauchy problem.

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## 1 Introduction

Fractional calculus has gained increasing attention during the past three decades due to its applications. It has been successfully applied to those fields that include computational biology, economics, physics and so on. See the monographs by Kilbas et al. [23], Podlubny [35], Diethelm [15], Zhou [42, 43], the papers [5, 6, 11, 12, 26, 36, 38] and the references therein. In recent years, discrete fractional calculus has received increasing interest by many mathematicians as well. Gray and Zhang [21] developed a fractional calculus for the discrete nabla difference operator. Atici and Eloe [7, 8] developed the delta type fractional

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sums/differences with Riemann-Liouville-like operators and began the study of initial value problems. The existence, uniqueness and positivity of solutions for the discrete fractional boundary value problems have been studied by Goodrich [17, 18], as well as monotonicity properties [14]. The modeling with fractional difference equations began to be studied by Atici and Sengül [9]. Ferreira [16] proved a discrete fractional Gronwall inequality. More recently, Wu and Baleanu [39] studied the discrete fractional logistic map and its chaos, notably with the same definition of fractional delta operator that we will consider in this paper. For an overview, one can see the recent monograph by Goodrich and Peterson [19].

In spite of the extensive research in this area, there are still some outstanding problems regarding fractional difference equations. In particular, further research of abstract fractional difference equations with unbounded linear operators remains to be done. Some models among these equations are closely connected with numerical methods for integro-partial differential equations that are intermediate between diffusion and wave equations [13] and evolution equations with memory [33]. On the other hand, the mixed models that come from time discretization of partial differential equations occur not only in traffic dynamics, but also in the theory of probability and in the theory of the chain processes of chemistry and radioactivity [10]. Thus, the modeling by means of abstract fractional difference equations provide a new viewpoint and should give new insights about time-discrete behavior because we are taking into account memory effects of the materials that are implicitly present in the mathematical modeling. Besides, in the reference [40] the authors have pointed out that discrete fractional models have some new degrees of freedom which can be used to capture the hidden aspects of real world phenomena with memory effects. Thus, it is worthwhile to study the behavior of abstract fractional difference equations not only from a purely mathematical but also an applied perspective.

As for the abstract fractional difference equations, some results were presented for the first time in [27, 28]. Since then, some developments have been made motivated by these researches [2–4, 22, 24, 25, 29–32]. In [28], by using an operator theoretical method, the author was successful to completely characterize the maximal regularity of solutions for a class of discrete time evolution equations. In [27], Lizama considered the existence and stability for fractional difference equations

$$\begin{cases} {}^C\Delta^\alpha u(n) = Au(n+1), \quad n \in \mathbb{N}_0, \\ u(0) = u_0 \in X, \end{cases} \tag{1.1}$$

where ${}^C\Delta^\alpha$ is the Caputo-like fractional difference operator of order $0 < \alpha \le 1$, and $A$ is a closed linear operator with domain $D(A)$ defined on a Banach space $X$, $\mathbb{N}_0 = \{0, 1, 2, ...\}$. It was the first time that the problem (1.1) showed a strong link between the fractional differential and fractional difference operators. Further, by applying the properties of Poisson transformation, a concept that was introduced in [27], as well as methods of operator theory, some results of continuous fractional evolution equations can be generalized to situations involving abstract fractional difference equations. In [1], Abadias and Lizama studied the existence and uniqueness of almost automorphic solutions for nonlinear partial difference-differential equations modeled in abstract form as

$$\Delta^\alpha u(n) = Au(n+1) + f(n, u(n)), \quad n \in \mathbb{Z},$$

for $0 < \alpha \leq 1$ where $A$ is the generator of a $C_0$-semigroup defined on a Banach space $X$ and $\Delta^\alpha$ denote the fractional difference in Weyl-like sense. Some recent works of fractional models with the Grünwald-Letnikov fractional difference can be found in [30, 37] and the references therein, these models can serve as a new microstructural basis for the fractional nonlocal continuum mechanics and physics.

Motivated by the above mentioned works, in this paper, we consider the existence of solutions for nonlinear abstract fractional difference equations

$$\begin{cases} \Delta^\alpha u(n) = Au(n+1) + f(n, u(n)), \quad n \in \mathbb{N}_0; \\ u(0) = u_0 \in X, \end{cases} \tag{1.2}$$

where $\Delta^\alpha$ is the Riemann-Liouville-like fractional difference operator of order $0 < \alpha \leq 1$, $f : \mathbb{N}_0 \to X$, $A$ is the infinitesimal generator of a bounded $C_0$-semigroup $\{Q(t)\}_{t \geq 0}$ with domain $D(A)$ defined on a Banach space $X$. It is important to remark that such problem has only been recently studied in [29] but with $A$ bounded. Therefore, our main contribution in this paper is a significative advance in the study of the Cauchy problem (1.2) with unbounded operator $A$, typically differential operators like the Laplacian, allowing in this way to the analysis of mixed fractional difference-differential equations, using tools of operator theory.

One reason to consider in this paper the Riemann-Liouville-like fractional difference operator arises in the recent paper [27] where is proved that $\Delta^\alpha$ is linked with the Riemann-Lioville fractional differential operator $D^\alpha$ by means of the Poisson transformation $\mathbb{P}$. More precisely, the following identity is true: $\Delta^\alpha \circ \mathbb{P} = \mathbb{P} \circ D^\alpha$. See [4, Theorem 4.2 and Theorem 4.5] for an up to date revision of the main properties of the Poisson transformation and a proof of the above mentioned identity. A second reason, is that several fractional difference operators appearing in the current literature are, in fact, related with the operator $\Delta^\alpha$. For instance, we have the identity ${}^C\Delta^\alpha u(n) = \Delta^\alpha u(n) - k^{1-\alpha}(n+1)u(0)$ where $0 < \alpha < 1$ and $k^{1-\alpha}$ is defined in (2.1) below. See [29, Theorem 2.4]. A second example is the identity $\Delta^\alpha \circ \tau_a = \tau_{a+1-\alpha} \circ \Delta_a^\alpha$ valid for $0 < \alpha < 1$ and $a \in \mathbb{R}$, where $\tau_a$ denotes the translation operator and $\Delta_a^\alpha$ is the fractional difference operator as defined by Atici and Eloe [8]. This remarkable relationship, known as a transference principle, has been recently proved [20].

We remark that, in contrast with the above mentioned difference operators, the operator $\Delta^\alpha$ enjoy many good properties that enables the handle of abstract fractional difference equations in a simpler way. The main properties are that $\Delta^\alpha$ is a well defined operator in the vector-valued sequence space $s(\mathbb{N}_0, X)$, and that behaves nicely under (finite) convolution [4]. Consequently, many useful tools, like the $z$-transform, are directly available.

This article is organized as follows. In Section 2, we introduce the main tools needed to this work. In Section 3 we prove that, under the assumption that $A$ generates a bounded $C_0$-semigroup $\{Q(t)\}_{t\geq 0}$, there exists a sequence of bounded and linear operators $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ that is related with the semigroup $\{Q(t)\}_{t\geq 0}$ by means of the subordination formula:

$$S_\alpha(n)x = \int_0^\infty \int_0^\infty p_n(t) f_{s,\alpha}(t) Q(s)x \, dsdt, \quad n \in \mathbb{N}_0, \ x \in X,$$

where $f_{s,\alpha}$ is the Lévy distribution and $p_n$ is the Poisson distribution. This result improves [1, Theorem 3.5]. We note that an important property of the sequence of operators $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ is its automatic regularity: $S_\alpha(n)x \in D(A)$ for all $n \in \mathbb{N}_0$. Using this remarkable fact, we show in Section 4 that $u : \mathbb{N}_0 \to [D(A)]$ verifies (1.2) if and only if $u$ satisfies $u(0) = u_0 \in D(A)$ and

$$u(n) = S_\alpha(n)(I - A)u_0 + \sum_{j=0}^{n-1} S_\alpha(n-1-j) f(j, u(j)), \ n \in \mathbb{N}.$$

In section 5, we present our main findings on the existence of solutions for abstract fractional semilinear difference equations modeled as (1.2). We prove that if $f$ satisfies a Lipschitz type condition, then for an initial condition $u_0 \in D(A)$ the problem (1.2) has a unique solution in the vector valued space of sequences $l^\infty(\mathbb{N}_0; X)$. Using a different argument involving the compactness of the semigroup $Q(t)$, we show that if $f$ is merely bounded in the first variable and sublinear in the second one, then the problem (1.2) with initial condition $u(0) = 0$ has at least one solution in the $fns$-space of sequences $l_f^\infty(\mathbb{N}; X)$. Note that such vector-valued Banach space was only recently introduced in the literature by Lizama and Velasco [29]. Finally, in Section 6, we provide some simple examples to illustrate our main findings.

## 2 Preliminaries

As usual, we denote $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $\mathbb{N} = \{1, 2, ...\}$. Let $X$ be a Banach space with norm $\|\cdot\|$, $\mathcal{B}(X)$ be the space of bounded linear operators from $X$ into $X$ endowed with the norm $\|Q\|_{\mathcal{B}(X)} = \sup\{\|Q(x)\| : \|x\| = 1\}$, where $x \in X$ and $Q \in \mathcal{B}(X)$. We denote by $s(\mathbb{N}_0; X)$ the vectorial space consisting of all vector-valued sequences $u : \mathbb{N}_0 \to X$.

In this context, the forward Euler operator $\Delta : s(\mathbb{N}_0; X) \to s(\mathbb{N}_0; X)$ is defined by

$$\Delta u(n) := u(n+1) - u(n), \quad n \in \mathbb{N}_0.$$

Recall that the finite convolution $*$ of two sequences $u, v \in s(\mathbb{N}_0; X)$ is defined as follows

$$(u * v)(n) := \sum_{j=0}^{n} u(n-j)v(j), \quad n \in \mathbb{N}_0.$$

In addition, for $\alpha > 0$, we consider the scalar sequence $\{k^\alpha(n)\}_{n\in\mathbb{N}_0}$ defined by

$$k^\alpha(n) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}, \quad n \in \mathbb{N}_0. \tag{2.1}$$

This scalar sequence was introduced by Lizama in [27, 28] in the context of fractional differences. It has several important properties. For instance, the semigroup property

$$(k^\alpha * k^\beta)(n) = \sum_{j=0}^{n} k^\alpha(n-j)k^\beta(j) = k^{\alpha+\beta}(n), \quad n \in \mathbb{N}_0,\ \alpha > 0,\ \beta > 0.$$

It is easy to see that for all $n \in \mathbb{N}_0$ and for any $\alpha \in (0, 1]$, $k^\alpha(n) \in (0, 1]$ and $k^\alpha(n)$ is a non-increasing sequence. Moreover, by [44, Vol I, p.77 (1.18)] we have

$$k^\alpha(n) = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + O\left(\frac{1}{n}\right)\right),\ n \in \mathbb{N},\ \alpha > 0. \tag{2.2}$$

**Definition 2.1.** [27] *Let $\alpha > 0$, the $\alpha$-th order fractional sum operator is defined by*

$$\Delta^{-\alpha}u(n) = \sum_{j=0}^{n} \frac{\Gamma(n-j+\alpha)}{\Gamma(\alpha)\Gamma(n-j+1)}u(j) = \sum_{j=0}^{n} k^\alpha(n-j)u(j), \quad n \in \mathbb{N}_0.$$

**Definition 2.2.** [27] *Let $\alpha > 0$, the $\alpha$-th order fractional difference operator* (in the sense of Riemann-Liouville-like) *is defined by*

$$\Delta^\alpha u(n) := \Delta^m \circ \Delta^{-(m-\alpha)}u(n), \quad n \in \mathbb{N}_0,$$

*where $m - 1 < \alpha < m$, $m = [\alpha] + 1$.*

**Lemma 2.1.** [28] *Let $\alpha \in (0, 1)$, $a : \mathbb{N}_0 \to \mathbb{C}$ and $b : \mathbb{N}_0 \to X$ be given. Then*

$$\Delta^\alpha(a * b)(n) = (a * \Delta^\alpha b)(n) + b(0)a(n+1), \quad n \in \mathbb{N}_0.$$

We recall the following classical result.

**Theorem 2.1.** (Schauder's Fixed Point Theorem). *Let $\Omega$ be a nonempty, closed and convex subset of Banach space $X$. If $P : \Omega \to \Omega$ is completely continuous, then $P$ has a fixed point in $\Omega$.*

Let us introduce the Mittag-Leffler function as follows

$$E_{\alpha,\beta}(z)=\sum_{n=1}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}=\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\mu^{\alpha-\beta}e^{\mu}}{\mu^{\alpha}-z}d\mu,\quad z,\beta\in\mathbb{C},\ Re(\alpha)>0,\tag{2.3}$$

where $\mathcal{C}$ is a contour which starts and ends at $-\infty$ and encircles the disc $|\mu|\leq|z|^{1/\alpha}$ counter-clockwise. Its Laplace transform is given by

$$\int_0^{\infty}e^{-\lambda t}t^{\beta-1}E_{\alpha,\beta}(\pm\omega t^{\alpha})dt=\frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha}\mp\omega},\quad \omega\in\mathbb{C},\ Re(\lambda)>|\omega|^{1/\alpha}.\tag{2.4}$$

We will need the following function, called stable Lévy process, which was introduced by Yosida [41]:

$$f_{t,\alpha}(\lambda)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}e^{z\lambda-tz^{\alpha}}dz,\quad \sigma>0,\ t>0,\ \lambda\geq 0,\ 0<\alpha<1,\tag{2.5}$$

where the branch of $z^{\alpha}$ is taken such that $Re(z^{\alpha})>0$ for $Re(z)>0$. This branch is single-valued in the $z$-plane cut along the negative real axis. We denote the kernel function

$$g_{\alpha}(t):=\frac{t^{\alpha-1}}{\Gamma(\alpha)},\quad t>0,\ \alpha>0,$$

and in case $\alpha=0$, we set $g_0(t)=\delta(t)$, the Dirac measure concentrated at the origin. It is well known that this function play a central role in the theory of fractional calculus. We well need the following result that gives insight on the relationship between fractional powers, the kernel function $g_{\alpha}$, the Mittag-Leffler function and stable Lévy processes.

**Proposition 2.1.** *The following properties hold:*

(i) $\int_0^{\infty}e^{-\lambda a}f_{t,\alpha}(\lambda)d\lambda=e^{-ta^{\alpha}}$, $t>0$, $a>0$.

(ii) $f_{t,\alpha}(\lambda)\geq 0$, $\lambda>0$.

(iii) $\int_0^{\infty}f_{t,\alpha}(\lambda)d\lambda=1$.

(iv) $\int_0^{\infty}f_{s,\alpha}(t)ds=g_{\alpha}(t)$, $t>0$.

(v) $\int_0^{\infty}e^{-\lambda s}f_{s,\alpha}(t)ds=t^{\alpha-1}E_{\alpha,\alpha,}(-\lambda t^{\alpha})$, $\lambda\in\mathbb{C}$, $t>0$.

*Proof.* The proof of (i)-(iii) can be found in [41, p.260-262]. Next, we shall (iv) holds. Since the function $f_{s,\alpha}(t)$ is non-negative for $t>0$, by applying (2.5) and the uniqueness of the inverse Laplace transform, we have

$$\int_0^{\infty}f_{s,\alpha}(t)ds=\frac{1}{2\pi i}\int_0^{\infty}\int_{\sigma-i\infty}^{\sigma+i\infty}e^{zt}e^{-sz^{\alpha}}dzds=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{1}{z^{\alpha}}e^{zt}dz=g_{\alpha}(t).$$

Next, we prove that (v) is true. In fact, for all $t > 0$, in view of (2.3) and $\lambda \in \mathbb{C}$, by applying (2.5) again, we have

$$\begin{aligned}\int_0^\infty e^{-\lambda s} f_{s,\alpha}(t)ds =& \frac{1}{2\pi i}\int_0^\infty \int_{\sigma-i\infty}^{\sigma+i\infty} e^{zt-sz^\alpha} e^{-\lambda s} dz ds \\ =& \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{zt}}{z^\alpha+\lambda} dz \\ =& t^{\alpha-1}\frac{1}{2\pi i}\int_{\sigma'-i\infty}^{\sigma'+i\infty} \frac{e^{z}}{z^\alpha+\lambda t^\alpha} dz = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha).\end{aligned}$$

Consequently, we obtain the desired results.

□

Finally, for each $n \in \mathbb{N}_0$, we recall that the Poisson distribution is defined by

$$p_n(t) := e^{-t}\frac{t^n}{n!}, \quad t \geq 0.$$

One of the most interesting properties is associated with their infinite integral

$$\int_0^\infty p_n(t)dt = 1, \quad n \in \mathbb{N}_0.$$

Moreover, we recall that the Poisson transformation for a continuous function $u(t)$, $t \in [0, \infty)$ is defined by

$$u(n) := \int_0^\infty p_n(t)u(t)dt, \ n \in \mathbb{N}_0. \tag{2.6}$$

This definition was introduced by Lizama [27]. As pointed out in [27], the Poisson transformation reveals a strong relation between $g_\alpha$ and $k^\alpha$, and consequently the fractional operators $D_t^\alpha$ and $\Delta^\alpha$.

## 3 Resolvent sequences

Our basic assumption in this section is that the operator $A$ in (1.2) is the infinitesimal generator of a bounded $C_0$-semigroup $\{Q(t)\}_{t\geq 0}$. This means that there is a constant $M \geq 1$ such that $M = \sup_{t\in[0,\infty)} \|Q(t)\|_{\mathcal{B}(X)} < \infty$. It is well known from [34, p.19, Theorem 5.2(i)] that $A$ is closed and $D(A)$ is dense in $X$. Next, we use the notion of $\alpha$-resolvent sequence of bounded and linear operators which was introduced by Abadias and Lizama [1] and it is an important tool to deal with abstract fractional difference equations.

**Definition 3.1.** *Let $\alpha > 0$ and $A$ be a closed linear operator with domain $D(A)$ defined on a Banach space $X$. An operator-valued sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ is called an $\alpha$-resolvent sequence generated by $A$ if it satisfies the following conditions*

(i) *$S_\alpha(n)x \in D(A)$ for all $x \in X$ and $AS_\alpha(n)x = S_\alpha(n)Ax$, for all $n \in \mathbb{N}_0$ and $x \in D(A)$;*

(ii) *$S_\alpha(n)x = k^\alpha(n)x + A(k^\alpha * S_\alpha)(n)x$, for all $n \in \mathbb{N}_0$ and $x \in X$.*

The main properties of $\alpha$-resolvent sequences are contained in the following result.

**Lemma 3.1.** [1] *Let $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ be an $\alpha$-resolvent sequence generated by $A$. Then*

(i) $1 \in \rho(A)$;

(ii) *For all $x \in X$ we have that $S_\alpha(0)x = (I-A)^{-1}x$ and there exists a scalar sequence $\{\beta_{\alpha,n}(j)\}_{n,j\in\mathbb{N}}$ such that*

$$S_\alpha(n)x = \sum_{j=1}^{n} \beta_{\alpha,n}(j)(I-A)^{-(j+1)}, \quad n \in \mathbb{N};$$

(iii) *For all $x \in X$ we have that $S_\alpha(0)x \in D(A)$ and $S_\alpha(n)x \in D(A^2)$ for all $n \in \mathbb{N}$.*

Our first result is an improvement of [1, Theorem 3.5] where the exponential stability of the semigroup is assumed.

**Theorem 3.1.** *Let $0 < \alpha \le 1$ and $A$ be the generator of a bounded $C_0$-semigroup $\{Q(t)\}_{t\ge0}$ defined on a Banach space $X$. Then, $A$ generates an $\alpha$-resolvent sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ given by*

$$S_\alpha(n)x = \int_0^\infty \int_0^\infty p_n(t) f_{s,\alpha}(t) Q(s)x \, dsdt, \quad n \in \mathbb{N}_0, \ x \in X. \tag{3.1}$$

*Proof.* From above assumption, we know that there exists $M \ge 1$ such that $\|Q(t)\|_{\mathcal{B}(X)} \le M$ for $t \ge 0$. Define

$$Q_\alpha(t)x := \int_0^\infty f_{s,\alpha}(t) Q(s)x ds, \quad t > 0, \ x \in X. \tag{3.2}$$

Firstly, we observe that $Q_\alpha(t)$ is well defined for $t > 0$. Indeed, in view of (2.5), Proposition 2.1 (ii) and (iv), for any $x \in X$, we have

$$\|Q_\alpha(t)x\| \le \int_0^\infty f_{s,\alpha}(t)\|Q(s)x\|ds \le M\|x\| \int_0^\infty f_{s,\alpha}(t)ds \le M\|x\| g_\alpha(t), \tag{3.3}$$

and hence, we obtain

$$\int_0^\infty e^{-Re(\lambda)t}\|Q_\alpha(t)x\|dt \le \frac{M\|x\|}{\Gamma(\alpha)} \int_0^\infty e^{-Re(\lambda)t} t^{\alpha-1} dt \le \frac{M\|x\|}{[Re(\lambda)]^\alpha},$$

for $Re(\lambda) > 0$, $x \in X$. Consequently, $Q_\alpha(t)$ is Laplace transformable and, using Fubini's theorem, we obtain

$$\widehat{Q}_\alpha(\lambda)x := \int_0^\infty e^{-\lambda t} Q_\alpha(t)x dt = \int_0^\infty \Big( \int_0^\infty e^{-\lambda t} f_{s,\alpha}(t) dt \Big) Q(s)x ds.$$

Therefore, by (i) in Proposition 2.1 and a well-known property on the Laplace transformation of $C_0$-semigroups, we have

$$(\lambda^\alpha I - A)\widehat{Q}_\alpha(\lambda)x = (\lambda^\alpha I - A)\int_0^\infty e^{-\lambda^\alpha s}Q(s)x\,ds = x, \quad x \in X,$$

and

$$\widehat{Q}_\alpha(\lambda)(\lambda^\alpha I - A)x = \int_0^\infty e^{-\lambda^\alpha s}Q(s)(\lambda^\alpha I - A)x\,ds = x, \quad x \in D(A).$$

It shows that $A$ commutes with $Q_\alpha(t)$ on $D(A)$ and

$$\widehat{Q}_\alpha(\lambda)x = \frac{1}{\lambda^\alpha}x + A\frac{1}{\lambda^\alpha}\widehat{Q}_\alpha(\lambda)x, \quad x \in X.$$

By the inversion of the Laplace transform, we obtain the identity

$$Q_\alpha(t)x = g_\alpha(t)x + A\int_0^t g_\alpha(t-s)Q_\alpha(s)xds, \quad x \in X, \tag{3.4}$$

and since $A$ is closed, we also get

$$Q_\alpha(t)x = g_\alpha(t)x + \int_0^t g_\alpha(t-s)Q_\alpha(s)Axds, \quad x \in D(A).$$

Since $A$ commutes with $Q_\alpha(t)$ and $A$ is closed, by applying Poisson transformation into (3.4), similarly to the remaining proof of [1, Theorem 3.5], one can see that

$$S_\alpha(n)x = k^\alpha(n)x + A\sum_{j=0}^{n} k^\alpha(n-j)S_\alpha(j)x, \ x \in X,$$

which implies that Definition 3.1 (i)-(ii) are satisfied. The proof is completed. □

The above theorem has two important consequences that we give as corollaries.

**Corollary 3.1.** *Let $0 < \alpha \le 1$ and $A$ be the generator of a bounded $C_0$-semigroup $\{Q(t)\}_{t\ge 0}$ defined on a Banach space $X$. Then,*

$$\|S_\alpha(n)x\| \le k^\alpha(n)\sup_{t\ge 0}\|Q(t)\|_{\mathcal{B}(X)}\|x\|, \quad \text{for } n \in \mathbb{N}_0, \ x \in X,$$

*where $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ is defined by (3.1).*

*Proof.* From Theorem 3.1, we know that $A$ generates an $\alpha$-resolvent sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$. Then for $M := \sup_{t\ge 0}\|Q(t)\|_{\mathcal{B}(X)}$ we have

$$\|S_\alpha(n)x\| \le \int_0^\infty\int_0^\infty p_n(t)f_{s,\alpha}(t)\|Q(s)x\|\,dsdt \le M\int_0^\infty e^{-t}\frac{t^n}{n!}g_\alpha(t)dt\|x\| = Mk^\alpha(n)\|x\|.$$

□

The second important consequence can be described as follows.

**Corollary 3.2.** *Let $0 < \alpha \leq 1$ and $A$ be the generator of a bounded and compact $C_0$-semigroup $\{Q(t)\}_{t>0}$ defined on a Banach space $X$. Then, $A$ generates a compact $\alpha$-resolvent sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ which is defined in (3.1).*

*Proof.* Let $\varepsilon > 0$ be given. We define an operator sequence $S_\alpha^\varepsilon$ as follows

$$\begin{aligned} S_\alpha^\varepsilon(n)x &:= \int_0^\infty \int_\varepsilon^\infty p_n(t) f_{s,\alpha}(t) Q(s)x \, dsdt \\ &= Q(\varepsilon) \int_0^\infty \int_\varepsilon^\infty e^{-t} \frac{t^n}{n!} f_{s,\alpha}(t) Q(s-\varepsilon)x \, dsdt, \end{aligned}$$

for $x \in X$, where in the second identity we use the semigroup property. Then, from the compactness of $Q(\varepsilon)(\varepsilon > 0)$, and from Corollary 3.1, we obtain that the set $V_\varepsilon = \{S_\alpha^\varepsilon(n)x : n \in \mathbb{N}_0\}$ is relatively compact in $X$ for $\varepsilon > 0$. Moreover, for any $x \in X$, we have

$$\begin{aligned} \|S_\alpha(n)x - S_\alpha^\varepsilon(n)x\| &= \left\| \int_0^\infty \int_0^\varepsilon e^{-t} \frac{t^n}{n!} f_{s,\alpha}(t) Q(s)x \, dsdt \right\| \\ &\leq M\|x\| \int_0^\varepsilon \int_0^\infty e^{-t} \frac{t^n}{n!} f_{s,\alpha}(t) \, dtds. \end{aligned}$$

Since $e^t > \frac{t^n}{n!}$ and $f_{s,\alpha}(t) \geq 0$ for $t > 0$, then $e^{-t}\frac{t^n}{n!} < 1$ and by (iii) in Proposition 2.1 we have

$$\int_0^\infty e^{-t} \frac{t^n}{n!} f_{s,\alpha}(t) dt < \int_0^\infty f_{s,\alpha}(t) dt = 1, \quad s > 0.$$

Therefore, we obtain

$$\int_0^\varepsilon \int_0^\infty e^{-t} \frac{t^n}{n!} f_{s,\alpha}(t) \, dtds < \epsilon.$$

Consequently,

$$\|S_\alpha(n)x - S_\alpha^\varepsilon(n)x\| \to 0, \quad \text{as } \varepsilon \to 0.$$

Hence, there are relatively compact sets arbitrarily close to the set $V = \{S_\alpha(n)x : n \in \mathbb{N}_0\}$ for $x \in X$. Thus, the set $V$ is also relatively compact in $X$. It means that the $\alpha$-resolvent sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ is compact. $\square$

Observe that the conclusion of compactness for $S_\alpha(n)$ also follows from the representation in Lemma 3.1 (ii), namely, assuming that $(I - A)^{-1}$ is a compact operator. In this way, no assumption on $A$ for the generation of a compact $C_0$-semigroup is needed.

## 4 The inhomogeneous Cauchy problem

In this section, we consider the inhomogeneous linear abstract fractional difference equations on a Banach space $X$ given by

$$\begin{cases} \Delta^\alpha u(n) = Au(n+1) + f(n), & n \in \mathbb{N}_0, \\ u(0) = u_0 \in X, \end{cases} \tag{4.1}$$

where $0 < \alpha \leq 1$ and $A$ is the generator of a bounded $C_0$-semigroup $\{Q(t)\}_{t\geq 0}$. Note that, in particular, $(I-A)^{-1}$ exists, because $\lambda \in \rho(A)$ for all $Re(\lambda) > 0$ by the Hille-Yosida theorem [34].

**Definition 4.1.** *Let $f \in s(\mathbb{N}_0; X)$ be given. We say that $u \in s(\mathbb{N}_0; X)$ is a strong solution of (4.1) if $u(n) \in D(A)$ for all $n \in \mathbb{N}_0$ and $u(n)$ satisfies (4.1).*

In what follows, we always denote by $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ the $\alpha$-resolvent sequence generated by $A$ which is defined in (3.1). The following theorem is the main result of this section. It is interesting to observe that, in contrast with the continuous case, no additional regularity on the range of the sequence $f$ is needed.

**Theorem 4.1.** *Let $0 < \alpha < 1$ and $f \in s(\mathbb{N}_0; X)$. Then (4.1) admits a strong solution $u \in s(\mathbb{N}_0; [D(A)])$ if and only if $u$ satisfies $u(0) = u_0 \in D(A)$ and*

$$u(n) = S_\alpha(n)(I-A)u_0 + \sum_{j=0}^{n-1} S_\alpha(n-1-j)f(j), \;\; n \in \mathbb{N}. \tag{4.2}$$

*Proof.* $\Leftarrow$) Suppose (4.2) holds. By (iii) in Lemma 3.1 we have $u(n) \in D(A)$ for all $n \in \mathbb{N}_0$. Next, we prove that the following identity holds:

$$\Delta^\alpha S_\alpha(n)x = AS_\alpha(n+1)x, \qquad n \in \mathbb{N}_0, \;\; x \in X.$$

Indeed, convolving the proposed identity of (ii) in Definition 3.1 by $k^{1-\alpha}$, we have

$$(k^{1-\alpha} * S_\alpha)(n)x = (k^{1-\alpha} * k^\alpha)(n)x + A(k^{1-\alpha} * k^\alpha * S_\alpha)(n)x, \;\; n \in \mathbb{N}_0.$$

Applying the semigroup property of the kernels $k^\alpha$, and the relationship between fractional sum and convolution, the above equality is equivalent to the following expression

$$\Delta^{-(1-\alpha)} S_\alpha(n)x = k^1(n)x + A\sum_{j=0}^{n} k^1(n-j)S_\alpha(j)x, \;\; n \in \mathbb{N}_0.$$

Therefore, using $k^1(j) = 1$ and $\Delta k^1(j) = 0$ for all $j \in \mathbb{N}_0$, we get

$$\begin{aligned}
\Delta^\alpha S_\alpha(n)x =& \Delta\Delta^{-(1-\alpha)} S_\alpha(n)x = \Delta k^1(n)x + A\Delta \sum_{j=0}^{n} k^1(n-j)S_\alpha(j)x \\
=& A\sum_{j=0}^{n+1} S_\alpha(j)x - A\sum_{j=0}^{n} S_\alpha(j)x \\
=& AS_\alpha(n+1)x,
\end{aligned}$$

for all $n \in \mathbb{N}_0$ and all $x \in X$, and then the claim is proved.

Next, we apply the operator $\Delta^\alpha$ into (4.2), it yields

$$\Delta^\alpha u(n) = \Delta^\alpha S_\alpha(n)(I-A)u_0 + \Delta^\alpha \sum_{j=0}^{n-1} S_\alpha(n-1-j)f(j), \quad n \in \mathbb{N}.$$

In addition, by Lemma 2.1, we have

$$\begin{aligned}
\Delta^\alpha(S_\alpha * f)(n-1) =& \sum_{j=0}^{n-1} \Delta^\alpha S_\alpha(j) f(n-1-j) + S_\alpha(0)f(n) \\
=& \sum_{j=1}^{n} \Delta^\alpha S_\alpha(j-1) f(n-j) + S_\alpha(0)f(n) \\
=& \sum_{j=1}^{n} A S_\alpha(j) f(n-j) + S_\alpha(0)f(n) \\
=& \sum_{j=0}^{n} A S_\alpha(j) f(n-j) - AS_\alpha(0)f(n) + S_\alpha(0)f(n) \\
=& A(S_\alpha * f)(n) + f(n),
\end{aligned}$$

where we applied the fact that $1 \in \rho(A)$ and the identity $S_\alpha(0)x = (I-A)^{-1}x$ for all $x \in X$, see Lemma 3.1. Therefore, it follows that

$$\begin{aligned}
\Delta^\alpha u(n) =& \Delta^\alpha S_\alpha(n)(I-A)u_0 + \Delta^\alpha(S_\alpha * f)(n-1) \\
=& AS_\alpha(n+1)(I-A)u_0 + \Delta^\alpha(S_\alpha * f)(n-1) \\
=& Au(n+1) - A(S_\alpha * f)(n) + \Delta^\alpha(S_\alpha * f)(n-1) \\
=& Au(n+1) - A(S_\alpha * f)(n) + A(S_\alpha * f)(n) + f(n) \\
=& Au(n+1) + f(n).
\end{aligned}$$

$\Rightarrow$) By hypothesis, $u(0) = u_0 \in D(A)$. Using the fact that $\Delta^\alpha S_\alpha(n) = AS_\alpha(n+1)$ and Lemma 2.1 we obtain

$$\Delta^\alpha(S_\alpha * u)(n-1) = A(S_\alpha * u)(n) + u(n), \quad n \in \mathbb{N}, \tag{4.3}$$

and again by Lemma 2.1,

$$\Delta^\alpha(S_\alpha * u)(n-1) = (S_\alpha * \Delta^\alpha u)(n-1) + S_\alpha(n)u(0), \quad n \in \mathbb{N}. \tag{4.4}$$

Therefore, if $u$ is a solution of (4.1), then

$$\begin{aligned}
(S_\alpha * \Delta^\alpha u)(n-1) =& \sum_{j=0}^{n-1} S_\alpha(n-1-j)\Delta^\alpha u(j) \\
=& \sum_{j=0}^{n-1} S_\alpha(n-1-j)Au(j+1) + \sum_{j=0}^{n-1} S_\alpha(n-1-j)f(j) \qquad (4.5)
\end{aligned}$$

$$
\begin{aligned}
&= A\sum_{j=0}^{n} S_\alpha(n-j)u(j) - AS_\alpha(n)u(0) + \sum_{j=0}^{n-1} S_\alpha(n-1-j)f(j)\\
&= A(S_\alpha * u)(n) - AS_\alpha(n)u(0) + (S_\alpha * f)(n-1).
\end{aligned}
$$

Putting (4.3) and (4.4) into (4.5), we obtain

$$
u(n) = S_\alpha(n)(I-A)u(0) + (S_\alpha * f)(n-1), \quad n \in \mathbb{N}.
$$

The proof is completed. □

## 5 The semilinear Cauchy problem

In this section, we study the existence of solutions for the following nonlinear abstract fractional difference equation

$$
\begin{cases}
\Delta^\alpha u(n) = Au(n+1) + f(n, u(n)), & n \in \mathbb{N}_0;\\
u(0) = u_0,
\end{cases} \tag{5.1}
$$

where $\alpha > 0$, $A$ is the generator of a bounded $C_0$-semigroup and $f : \mathbb{N}_0 \times X \to X$ is given.

We introduce the next definition of solutions.

**Definition 5.1.** *Let $0 < \alpha \leq 1$ and $A$ be the generator of an $\alpha$-resolvent sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$. We say that $u \in s(\mathbb{N}_0; [D(A)])$ is a solution of (5.1) if $u$ satisfies $u(0) = u_0 \in D(A)$ and*

$$
u(n) = S_\alpha(n)(I-A)u_0 + \sum_{j=0}^{n-1} S_\alpha(n-1-j)f(j, u(j)), \quad n \in \mathbb{N}. \tag{5.2}
$$

According to Theorem 4.1, this definition is consistent with true solutions of (5.1). Also note that $u(n) \in D(A)$ for all $n \in \mathbb{N}_0$ because we always have $S_\alpha(n)x \in D(A)$ for all $x \in X$ and $n \in \mathbb{N}_0$. See Lemma 3.1.

Firstly, we consider the problem (5.1) on the vector-valued Banach space of sequences $l^\infty(\mathbb{N}_0; X)$, which consists of the following set

$$
l^\infty(\mathbb{N}_0; X) := \{u : \ \mathbb{N}_0 \to X, \ \sup_{n\in\mathbb{N}_0} \|u(n)\| < \infty\},
$$

endowed with the norm $\|u\|_\infty = \sup_{n\in\mathbb{N}_0} \|u(n)\|$.

In order to state our first existence result, we will need the following hypothesis.

(H1) $A$ is the generator of a bounded $C_0$-semigroup $\{Q(t)\}_{t\geq 0}$ with $M := \sup_{t\geq 0} \|Q(t)\|_{\mathcal{B}(X)}$ and $\alpha$-resolvent sequence defined in (3.1) for $0 < \alpha < 1$.

(H2) $f(n,0)\equiv 0$, and there exist $\beta\in[\alpha,1)$ and $0<L<\frac{1}{M}$, such that

$$\|f(n,x)-f(n,y)\|\le Lk^{1-\beta}(n)\|x-y\|,\ \text{for any } x,y\in X,\ n\in\mathbb{N}_0.$$

Noting that, by (2.2), the hypothesis (H2) implies $f(n,x)\to 0$ as $n\to\infty$ for all $x\in X$.

The next is the first main theorem of this section, concerning bounded solutions of problem (5.1).

**Theorem 5.1.** *Assume that $A$ satisfies* (H1) *and $f$ satisfies* (H2)*. Then for any $u_0\in D(A)$, the problem (5.1) has a unique solution $u$ in $l^\infty(\mathbb{N}_0;X)$.*

*Proof.* Let us define the map $P: l^\infty(\mathbb{N}_0;X)\to l^\infty(\mathbb{N}_0;X)$ as follows

$$(Pu)(n):=S_\alpha(n)(I-A)u_0+\sum_{j=0}^{n-1}S_\alpha(n-1-j)f(j,u(j)),\ n\in\mathbb{N},$$

and $(Pu)(0)=u_0$. We first show that $P$ is well defined. In fact, let $u\in l^\infty(\mathbb{N}_0;X)$ be given, it follows from (H2) that

$$\|f(n,u(n))\|\le Lk^{1-\beta}(n)\|u(n)\|,\ \text{for any } u\in l^\infty(\mathbb{N}_0;X),\quad n\in\mathbb{N}_0.$$

By (H2), Corollary 3.1 and observing that $1+\alpha-\beta\in(0,1]$ for $0<\alpha\le\beta<1$, we obtain

$$\begin{aligned}\|(Pu)(n)\|\le&\|S_\alpha(n)(I-A)u_0\|+\sum_{j=0}^{n-1}\|S_\alpha(n-1-j)f(j,u(j))\|\\ \le&Mk^\alpha(n)\|(I-A)u_0\|+\sum_{j=0}^{n-1}\|S_\alpha(n-1-j)(f(j,u(j))-f(j,0))\|\\ \le&\|(I-A)u_0\|+ML\sum_{j=0}^{n-1}k^\alpha(n-1-j)k^{1-\beta}(j)\|u(j)\|\\ \le&\|(I-A)u_0\|+ML\|u\|_\infty k^{1+\alpha-\beta}(n-1)\\ \le&\|(I-A)u_0\|+ML\|u\|_\infty,\quad n\in\mathbb{N},\end{aligned}$$

where we used the fact that $k^s(n)<k^s(0)=1$ because $k^s(n)$ is a nonincreasing sequence for each $0<s\le 1$ and $n\in\mathbb{N}$. It implies that $P$ is well defined.

For any $u,v\in l^\infty(\mathbb{N}_0;X)$, by (H2), we get,

$$\begin{aligned}\|(Pu)(n)-(Pv)(n)\|\le&M\sum_{j=0}^{n-1}k^\alpha(n-1-j)\|f(j,u(j))-f(j,v(j))\|\\ \le&ML\sum_{j=0}^{n-1}k^\alpha(n-1-j)k^{1-\beta}(j)\|u(j)-v(j)\|\\ \le&MLk^{1+\alpha-\beta}(n-1)\|u-v\|_\infty\end{aligned}$$

$$\leq ML\|u-v\|_\infty,$$

for all $n \in \mathbb{N}$. Hence, it follows that $\|Pu - Pv\|_\infty \leq ML\|u-v\|_\infty$. In view of $ML < 1$, we conclude the result by the Banach fixed point theorem.

□

It order to study solutions that behaves like $nn!$ at infinity, we consider the $fns$-space of vector-valued sequences $l_f^\infty(\mathbb{N}; X)$ introduced in [29] and defined by

$$l_f^\infty(\mathbb{N}; X) := \left\{ u : \mathbb{N} \to X, \ \sup_{n\in\mathbb{N}} \frac{\|u(n)\|}{nn!} < \infty \right\},$$

endowed with their natural norm $\|u\|_f = \sup\limits_{n\in\mathbb{N}} \dfrac{\|u(n)\|}{nn!}$. From [29] we note that the sequence $\dfrac{1}{nn!}\sum\limits_{j=0}^{n-1} jj!$ has the following properties

$$\sup_{n\in\mathbb{N}} \frac{1}{nn!}\sum_{j=0}^{n-1} jj! = \frac{5}{18} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{nn!}\sum_{j=0}^{n-1} jj! = \lim_{n\to\infty} \frac{n!-1}{nn!} = 0. \tag{5.3}$$

We will need the following Lemma.

**Lemma 5.1.** [29] *Let $U \subset l_f^\infty(\mathbb{N}; X)$ be such that*

(a) *The set $H_n(U) = \left\{ \dfrac{u(n)}{nn!} : \ u \in U \right\}$ is relatively compact in $X$, for all $n \in \mathbb{N}$.*

(b) $\lim\limits_{n\to\infty} \sup\limits_{u\in U} \dfrac{\|u(n)\|}{nn!} = 0$, *that is, for each $\varepsilon > 0$, there is $N > 0$ such that $\dfrac{\|u(n)\|}{nn!} < \varepsilon$, for each $n \geq N$ and for all $u \in U$.*

*Then $U$ is relatively compact in $l_f^\infty(\mathbb{N}; X)$.*

For a given function $g : \mathbb{N}_0 \times X \to X$, the Nemytskii operator $N_g : l_f^\infty(\mathbb{N}; X) \to l_f^\infty(\mathbb{N}; X)$ (with $g$ restricted to $\mathbb{N}$) is defined by

$$N_g(u)(n) := g(n, u(n)), \quad n \in \mathbb{N}.$$

In order to obtain our second main result, we will need the following assumptions

(H3) $A$ is the generator of a compact $C_0$-semigroup $\{Q(t)\}_{t>0}$ and $\alpha$-resolvent sequence defined in (3.1) for $0 < \alpha < 1$.

(H4) There exists a positive sequence $a(\cdot) \in l^\infty(\mathbb{N}_0)$ and a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\psi(r) \leq r$ for $r \in \mathbb{R}_+$ such that $\|g(n,x)\| \leq a(n)\psi\big(\|x\|\big)$, for all $n \in \mathbb{N}_0$ and $x \in X$.

(H5) The Nemytskii operator $N_g$ is continuous in $l_f^\infty(\mathbb{N}; X)$.

**Example 5.1.** *It is easy to check that the function $g : \mathbb{N}_0 \times X \to X$ defined by $g(n, x) := \cos(n)x$ satisfies* (H4) *and* (H5)*.*

**Theorem 5.2.** *Assume that $A$ satisfies* (H3) *and $g$ satisfies* (H4)-(H5)*. Then, the problem (5.1) with $u(0) = 0$ has at least one solution $u \in l^\infty_f(\mathbb{N}; X)$ provided that $\|a\|_\infty \leq 18/\big(5 \sup_{t\geq 0} \|Q(t)\|_{\mathcal{B}(X)}\big)$.*

*Proof.* Let us define the map $\mathcal{P} : l^\infty_f(\mathbb{N}; X) \to l^\infty_f(\mathbb{N}; X)$ as follows

$$(\mathcal{P}u)(n) := S_\alpha(n-1)g(0, u(0)) + \sum_{j=1}^{n-1} S_\alpha(n-1-j)g(j, u(j)),\ \ n \in \mathbb{N}.$$

Since $u(0) = 0$ and by hypothesis (H4), we have $g(n, 0) = 0$ for all $n \in \mathbb{N}_0$. Then we can rewrite $\mathcal{P}$ as

$$(\mathcal{P}u)(n) = \sum_{j=0}^{n-1} S_\alpha(n-1-j)g(j, u(j)),\ \ n \in \mathbb{N},$$

where we understand $u$ as their extension to $\mathbb{N}_0$ by $u(0) = 0$. Firstly, we show that $\mathcal{P}$ is well defined. Let $u \in l^\infty_f(\mathbb{N}; X)$ be given. For each $n \in \mathbb{N}$, by Corollary 3.1 and (H4), we have

$$\begin{aligned}
\|(\mathcal{P}u)(n)\| \leq& \sum_{j=0}^{n-1} \|S_\alpha(n-1-j)g(j, u(j))\| \\
\leq& \sup_{t\geq 0} \|Q(t)\|_{\mathcal{B}(X)} \sum_{j=0}^{n-1} k^\alpha(n-1-j)a(j)\psi\big(\|u(j)\|\big) \\
\leq& \sup_{t\geq 0} \|Q(t)\|_{\mathcal{B}(X)}\|a\|_\infty \sum_{j=0}^{n-1} \|u(j)\| \\
\leq& \sup_{t\geq 0} \|Q(t)\|_{\mathcal{B}(X)}\|a\|_\infty\|u\|_f \sum_{j=0}^{n-1} jj!,
\end{aligned}$$

where we used the fact that $k^\alpha(n) < k^\alpha(0) = 1$ because $k^\alpha(n)$ is a non increasing sequence for $0 < \alpha \leq 1$ and $n \in \mathbb{N}$. Denote $M := \sup_{t\geq 0} \|Q(t)\|_{\mathcal{B}(X)}$. Then, by (5.3) we obtain

$$\frac{\|(\mathcal{P}u)(n)\|}{nn!} \leq M\|a\|_\infty\|u\|_f \frac{1}{nn!} \sum_{j=0}^{n-1} jj! \leq \frac{5}{18} M\|a\|_\infty\|u\|_f. \tag{5.4}$$

This proves that $\mathcal{P}$ is well defined. Now, we show that $\mathcal{P}$ is continuous. Let $\{u_m\}_{m=1}^\infty \subset l^\infty_f(\mathbb{N}; X)$ be a sequence such that $u_m \to u$ as $m \to \infty$ in the norm of $l^\infty_f(\mathbb{N}; X)$. Then

$$\|(\mathcal{P}u_m)(n) - (\mathcal{P}u)(n)\| \leq M \sum_{j=0}^{n-1} k^\alpha(n-1-j)\|g(j, u_m(j)) - g(j, u(j))\|$$

$$\begin{aligned}
&\leq M\sum_{j=0}^{n-1}\|g(j,u_m(j))-g(j,u(j))\| \\
&\leq M\|N_g(u_m)-N_g(u)\|_f\sum_{j=0}^{n-1}jj!.
\end{aligned}$$

Therefore, for all $n\in\mathbb{N}$, we have

$$\begin{aligned}
\frac{\|(\mathcal{P}u_m)(n)-(\mathcal{P}u)(n)\|}{nn!} &\leq M\|N_g(u_m)-N_g(u)\|_f\frac{1}{nn!}\sum_{j=0}^{n-1}jj! \\
&\leq\frac{5}{18}M\|N_g(u_m)-N_g(u)\|_f\to 0, \quad \text{as } m\to\infty,
\end{aligned}$$

which means that $\|\mathcal{P}u_m-\mathcal{P}u\|_f\to 0$ as $m\to\infty$. Therefore $\mathcal{P}$ is continuous.

Since $Q(t)$ is compact for $t>0$, then from Corollary 3.2, we know that the sequence of operators $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ is compact. Let $r>0$ be given. We define a set by

$$\mathcal{S}_r:=\{\omega\in l_f^\infty(\mathbb{N};X)|\ \|\omega\|_f\leq r\}.$$

Clearly, $\mathcal{S}_r$ is a nonempty, bounded, closed and convex subset of $l_f^\infty(\mathbb{N};X)$. In view of (5.4) and (H4), we can deduce that $\mathcal{P}$ maps $\mathcal{S}_r$ into itself. Thus, it remains to show that $\mathcal{P}$ is a compact operator.

In order to prove that $U:=\mathcal{P}\mathcal{S}_r$ is relatively compact, we will use Lemma 5.1. We check that the conditions in this lemma are satisfied:

(a) Let $v=\mathcal{P}u$ for any $u\in\mathcal{S}_r$. We have

$$\begin{aligned}
v(n)=(\mathcal{P}u)(n) &=\sum_{k=0}^{n-1}S_\alpha(n-1-k)g(k,u(k)) \\
&=\sum_{k=0}^{n-1}S_\alpha(k)g(n-1-k,u(n-1-k)),\ n\in\mathbb{N},
\end{aligned}$$

and then,

$$\frac{v(n)}{nn!}=\frac{1}{n!}\left(\frac{1}{n}\sum_{k=0}^{n-1}S_\alpha(k)g(n-1-k,u(n-1-k))\right).$$

Therefore $\frac{v(n)}{nn!}\in\frac{1}{n!}co(K_n)$, where $co(K_n)$ denotes the convex hull of $K_n$ for the set

$$K_n=\bigcup_{k=0}^{n-1}\Big\{S_\alpha(k)g(\xi,x):\xi\in\{0,1,2,...,n-1\},\|x\|_f\leq r\Big\},\ n\in\mathbb{N}.$$

On the one hand, for every $m\in\mathbb{N}_0$ and $\sigma>0$, the set $\{g(k,x):\ 0\leq k\leq m,\ \|x\|_f\leq\sigma\}$ is bounded because from condition (H4) we have $\|g(k,x)\|\leq a(k)\psi\big(\|x(k)\|\big)\leq mm!\|a\|_\infty\sigma$ for all $0\leq k\leq m$ and $\|x\|_f\leq\sigma$. Consequently, the set $\{S_\alpha(n)g(k,x)\ :\ 0\leq k\leq$

$m,\ \|x\|_f \le \sigma\}$ is relatively compact in $X$ for all $n \in \mathbb{N}_0$ from the fact that $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ is compact. Then it follows that each set $K_n$ is relatively compact. From the inclusions $H_n(U) = \left\{\frac{v(n)}{nn!} : v \in U\right\} \subseteq \frac{1}{n!}co(K_n) \subseteq \frac{1}{n!}co(\overline{K_n})$, we conclude that the set $H_n(U)$ is relatively compact in $X$, for all $n \in \mathbb{N}$.

(b) Let $u \in \mathcal{S}_r$ and $v = \mathcal{P}u$. Using condition (H4), for each $n \in \mathbb{N}$ we have

$$\begin{aligned}\frac{\|v(n)\|}{nn!} \le & \frac{1}{nn!}\sum_{j=0}^{n-1}\|S_\alpha(n-1-j)g(j,u(j))\| \\ \le & \frac{M}{nn!}\|a\|_\infty\sum_{j=0}^{n-1}k^\alpha(n-1-j)\|u(j)\| \\ \le & M\|u\|_f\|a\|_\infty\frac{1}{nn!}\sum_{j=0}^{n-1}jj! \le Mr\|a\|_\infty\frac{1}{nn!}\sum_{j=0}^{n-1}jj!.\end{aligned}$$

Then (5.3) implies that $\lim_{n\to\infty}\frac{\|v(n)\|}{nn!} = 0$ independently of $u \in \mathcal{S}_r$. Therefore, $U = \mathcal{P}\mathcal{S}_r$ is relatively compact in $l_f^\infty(\mathbb{N}; X)$ from Lemma 5.1, and by applying the continuity of operator $\mathcal{P}$, we conclude that $\mathcal{P}$ is a completely continuous operator. Thus, the Schauder's fixed point theorem shows that $\mathcal{P}$ has at least one fixed point $u \in l_f^\infty(\mathbb{N}; X)$. The proof is completed. □

## 6 Examples

**Example 6.1.** *We can describe the heat flow in a ring of length one with a temperature dependent "source" at discrete time $n \in \mathbb{N}$ by the following evolution equation (see [34, p.234] and references therein)*

$$\begin{cases} u(n,z) - u(n-1,z) = \dfrac{\partial^2}{\partial z^2}u(n,z) + G(u(n,z)), & 0 < z < 1, \\ u(n-1,0) = u(n-1,1),\ u_z'(n-1,0) = u_z'(n-1,1), & \\ u(0,z) = u_0(z), & \end{cases} \tag{6.1}$$

*where $G$ is a given function.*

We rewrite this model as an abstract difference equation. As a natural Banach space, we choose $X = C_p([0,1])$ the space of all continuous real valued periodic functions having period 1 with the norm $\|u\| = \max_{0\le z\le 1}|u(z)|$. The space $X$ consists therefore of continuous functions on $[0,1]$ satisfying $u(0) = u(1)$.

On this Banach space $X$, we define an operator $A$ by $Av = v''$ with its domain

$$D(A) = \{v :\ v,\ v',\ v'' \in X,\ v(0) = v(1)\}.$$

Then, by [34, Chapter 8, Lemma 2.1], the operator $A$ generates a $C_0$-semigroup $\{Q(t)\}_{t\geq 0}$ which is compact, bounded and analytic on $X$.

With those definitions, Equation (6.1) is a particular case of the abstract difference equation

$$\Delta^{\alpha}x(n) = Ax(n+1) + f(n, x(n)), \quad n \in \mathbb{N}_0, \tag{6.2}$$

where $0 < \alpha \leq 1$, $x : \mathbb{N}_0 \to X$ is defined by $x(n)(z) := u(n, z)$ and the function $f : \mathbb{N}_0 \times X \to X$ is given by

$$f(n, x(n))(z) = G(u(n, z)).$$

In particular, if $f = 0$, then the solution of (6.2) with initial condition $x(0) = x_0$ and $\alpha = 1$ is given by

$$x(n) = (I - A)^{-n}x_0, \quad n \in \mathbb{N}.$$

Hence, by Theorem 3.1, we conclude that for each $0 < \alpha \leq 1$, the operator $A$ generates a $\alpha$-resolvent sequence $S_{\alpha}(n)$. By Corollary 3.2, the operator $A$ generates a compact $\alpha$-resolvent sequence $S_{\alpha}(n)$.

**Example 6.2.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $X = L^2(\Omega)$. We consider the following discrete abstract Cauchy problem*

$$\begin{cases} \Delta^{\alpha}u(n, z) = \nabla^2 u(n+1, z) + f(n, z), & n \in \mathbb{N}_0, \ z \in \Omega, \\ u(n, z) = 0, \quad n \in \mathbb{N}_0, \ z \in \partial\Omega \\ u(0, z) = u_0(z), \ z \in \Omega, \end{cases} \tag{6.3}$$

*where $f : \mathbb{R}^+ \times X \to X$ and $\nabla^2$ is the Laplacian operator.*

Now, let $A = \nabla^2$ be the Laplacian operator with Dirichlet boundary conditions and

$$D(A) = \left\{ v \in H_0^1(\Omega) \cap H^2(\Omega), \ Av \in L^2(\Omega) \right\}.$$

We denote by $\{-\lambda_l, \phi_l\}_{l=1}^{\infty}$ the eigensystem of the operator $A$, where $\{\lambda_l\}_{l=1}^{\infty}$ denotes the set of eigenvalues satisfying $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_l \leq \cdots$, and $\lambda_l \to \infty$ as $l \to \infty$, and $\{\phi_l\}_{l=1}^{\infty}$ denotes the corresponding eigenfunctions. It is well know that $\phi_l$ can be normalized so that $\{\phi_l\}_{l=1}^{\infty}$ is an orthonormal basis of $X$. Hence, it yields

$$Au = -\sum_{l=1}^{\infty} \lambda_l (u, \phi_l)\phi_l, \quad u \in D(A),$$

where $(\cdot, \cdot)$ is the inner product in $X$. It is clear that the operator $A$ generates a $C_0$-semigoup $\{Q(t)\}_{t>0}$ which is compact, bounded, analytic and explicitly given by

$$Q(t)u = \sum_{l=1}^{\infty} e^{-\lambda_l t}(u, \phi_l)\phi_l, \quad u \in D(A).$$

Hence, by applying Theorem 3.1, (v) in Proposition 2.1 and Corollary 3.2, we can obtain a subordinated, discrete compact $\alpha$-resolvent family $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ as follows

$$\begin{aligned} S_\alpha(n)u &= \int_0^\infty \int_0^\infty p_n(t) \sum_{l=1}^\infty f_{s,\alpha}(t) e^{-\lambda_l s}(u,\phi_l)\phi_l ds dt \\ &= \int_0^\infty p_n(t) \sum_{l=1}^\infty t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_l t^\alpha)(u,\phi_l)\phi_l dt. \end{aligned} \tag{6.4}$$

By applying [44, Theorem 5.1] and (2.4), we have

$$\begin{aligned} \int_0^\infty p_n(t) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_l t^\alpha) dt =& \frac{(-1)^n}{n!} \left(\mathcal{L}\left(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_l t^\alpha)\right)\right)^{(n)}(1) \\ =& \frac{(-1)^n}{n!} \left((s^\alpha+\lambda_l)^{-1}\right)^{(n)}\Big|_{s=1}, \end{aligned}$$

where $\mathcal{L}$ denotes the Laplace transform. Let $\beta_{\alpha,0}(0) := 1$ and $\beta_{\alpha,n}(0) := 0$ for $n \in \mathbb{N}$, then by (ii) in Lemma 3.1, there exists a scalar sequence $\{\beta_{\alpha,n}(j)\}_{n,j\in\mathbb{N}_0}$ such that

$$\frac{(-1)^n}{n!} \left((s^\alpha+\lambda_l)^{-1}\right)^{(n)}\Big|_{s=1} = \sum_{j=0}^n \beta_{\alpha,n}(j)(1+\lambda_l)^{-(j+1)}, \quad n \in \mathbb{N}_0.$$

Thus, we have

$$S_\alpha(n)u = \sum_{l=1}^\infty \sum_{j=0}^n \beta_{\alpha,n}(j)(1+\lambda_l)^{-(j+1)}(u,\phi_l)\phi_l, \quad u \in D(A),\ n \in \mathbb{N}_0. \tag{6.5}$$

Therefore, we obtain that (6.3) possesses a solution and its explicit form is given by

$$\begin{aligned} u(n) =& \sum_{l=1}^\infty \sum_{j=0}^n \beta_{\alpha,n}(j)(1+\lambda_l)^{-j}(u_0,\phi_l)\phi_l \\ &+ \sum_{m=0}^{n-1} \sum_{l=1}^\infty \sum_{j=0}^{n-1-m} \beta_{\alpha,n-1-m}(j)(1+\lambda_l)^{-(j+1)}(f(m,\cdot),\phi_l)\phi_l,\ n \in \mathbb{N}. \end{aligned}$$

**Example 6.3.** *For any $\lambda \in \mathbb{C}$, $|\lambda| < 1$ and $0 < \alpha \le 1$, we consider the following equations*

$$\begin{cases} \Delta^\alpha u(n) = -\lambda u(n+1) + f(n,u(n)), \\ u(0) = u_0. \end{cases} \tag{6.6}$$

It is clear that $Re(\lambda) > 0$ is the generator of the bounded $C_0$-semigroup $Q(t) = e^{-\lambda t}$ for $t \ge 0$. Hence, by applying Theorem 3.1, (v) in Proposition 2.1 and the definition of Mittag-Leffler function, we obtain the discrete $\alpha$-resolvent family $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ as follows

$$\begin{aligned} S_\alpha(n) &= \int_0^\infty \int_0^\infty p_n(t) f_{s,\alpha}(t) e^{-\lambda s} ds dt \\ &= \int_0^\infty p_n(t) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) dt = \sum_{i=0}^\infty (-\lambda)^i \frac{\Gamma(\alpha i+\alpha+n)}{\Gamma(\alpha i+\alpha)\Gamma(n+1)}. \end{aligned}$$

Then, by the definition of $k^\alpha(n)$, the explicit form of the solution of equation (6.6) is given by

$$u(n) = \sum_{i=0}^{\infty}(-\lambda)^i(1-\lambda)k^{\alpha i+\alpha}(n)u_0 + \sum_{i=0}^{\infty}\sum_{j=0}^{n-1}(-\lambda)^i k^{\alpha i+\alpha}(n-j-1)f(j,u(j)),\ n \in \mathbb{N}.$$

**Example 6.4.** *We consider the following nonlinear abstract fractional difference equations*

$$\begin{cases} \Delta^{0.5}x(n) = Ax(n+1) + 0.1k^{0.4}(n)\sin(x(n)), & n \in \mathbb{N}_0, \\ x(0) = u_0. \end{cases} \tag{6.7}$$

If $A$ is the generator of a contraction $C_0$-semigroup $\{Q(t)\}_{t\geq 0}$, then by Theorem 3.1, we conclude that for each $0 < \alpha \leq 1$, the operator $A$ generates an $\alpha$-resolvent sequence $S_\alpha(n)$ with $M = 1$. Hence, (H1) holds. Denote $\alpha = 0.5$ and $f(n, x(n)) = 0.1k^{0.4}(n)\sin(x(n))$ for $n \in \mathbb{N}_0$. Note that

$$\begin{aligned} \|f(n,x(n)) - f(n,y(n))\| =& 0.1k^{0.4}(n)\|\sin(x(n)) - \sin(y(n))\| \\ =& 0.2k^{0.4}(n)\left\|\cos\left(\frac{x(n)+y(n)}{2}\right)\sin\left(\frac{x(n)-y(n)}{2}\right)\right\| \\ \leq& 0.1k^{0.4}(n)\|x(n)-y(n)\|, \end{aligned}$$

which implies that (H2) holds. Following Theorem 5.1, we obtain that for any $u_0 \in D(A)$ there exist a unique solution of (6.7) in $l^\infty(\mathbb{N}_0; X)$.

If $A$ is the generator of a compact $C_0$-semigroup $\{Q(t)\}_{t>0}$, then by Corollary 3.2, we conclude that for each $0 < \alpha \leq 1$, the operator $A$ generates a compact $\alpha$-resolvent sequence $S_\alpha(n)$. Then (H3) holds. Denote $\alpha = 0.5$ and $g(n, x(n)) = k^{0.8}(n)\sin(x(n))$ for $n \in \mathbb{N}_0$. Consider the Nemystkii operator $N_g(u) : \mathbb{N} \to X$ defined by $N_g(u)(n) := g(n, u(n))$. Obviously,

$$\|g(n,x(n))\| = a(n)\|\sin(x(n))\| \leq a(n)\|x(n)\|,$$

where $a(n) = k^{0.8}(n)$. Then (H4) and (H5) hold. Thus, following Theorem 5.2, we obtain the existence of at least one solution of

$$\begin{cases} \Delta^{0.5}x(n) = Ax(n+1) + k^{0.8}(n)\sin(x(n)), & n \in \mathbb{N}_0, \\ x(0) = 0. \end{cases}$$

in the space $l_f^\infty(\mathbb{N}; X)$.

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